Centre for Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

tuesday, 26-06-2007, room 5118.-149, 9.00-12.00

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

The Riemann tensor in N spacetime dimensions has components $R^d{}_{abc}$. We define $R_{abcd} = g_{ae}R^e{}_{bcd}$. The Riemann tensor satisfies the identities

$$R_{abcd} = -R_{bacd} = -R_{abdc}\,, \qquad (1)$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0\,. \qquad (2)$$

(1.1) Show, by using equations (1) and (2), that

$$R_{abcd} = R_{cdab}\,. \qquad (3)$$

From now on we assume that the number of spacetime dimensions N is equal to 3.

(1.2) Show that for $N = 3$ the Riemann tensor can be expressed in terms of the Ricci tensor as follows:

$$R^{ab}{}_{cd} = -\frac{1}{g}\epsilon^{abe}\epsilon_{cdf}(R^f{}_e - \tfrac{1}{2}\delta^f{}_e R)\,, \qquad (4)$$

where

$$R^{ab}{}_{cd} = g^{be}R^a{}_{ecd}\,, \qquad (5)$$

$$g = \det\,(g_{ab})\,, \qquad (6)$$

and ϵ^{abc} is the completely anti-symmetric Levi-Civita symbol in 3 dimensions ($\epsilon^{012} = 1$).

Hint: Use the identities (without proof)

$$\epsilon^{acd}\epsilon_{bef}R^{ef}{}_{cd} = -4g(R^a{}_b - \tfrac{1}{2}\delta^a{}_b R)\,, \quad (7)$$
$$\epsilon^{abe}\epsilon_{cde} = g(\delta^a{}_c\delta^b{}_d - \delta^b{}_c\delta^a{}_d)\,. \quad (8)$$

(1.3) Show that the result of question (1.2) implies that every solution of the Einstein equation with $T_{ab} = 0$ describes a flat space.

(1.4) Take $T_{ab} = \Lambda g_{ab}$, with Λ is constant. Show that every solution of the Einstein equation corresponds to a maximally symmetric space.

Question 2

The Maxwell equations in a four-dimensional curved space can be written in the form

$$\nabla_\nu F^{\mu\nu} = j^\mu\,, \quad (9)$$
$$\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0\,. \quad (10)$$

Here F is the anti-symmetric field-strenght tensor and j the current. The covariant derivatives are with respect to the metric connection.

(2.1) Show that equation (10) for $F_{\mu\nu}$ is equivalent to

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0\,. \quad (11)$$

Hint: The covariant derivative of a contravariant vector V^μ and a covariant vector W_μ are given by

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda\,,$$
$$\nabla_\nu W_\mu = \partial_\nu W_\mu - \Gamma^\lambda_{\mu\nu} W_\lambda\,. \quad (12)$$

(2.2) We write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{13}$$

where A_μ is a covariant vector. Show that the $F_{\mu\nu}$ defined in this way is a covariant tensor by showing that

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{14}$$

Show that this $F_{\mu\nu}$ is a solution of equation (10).

(2.3) Show that equation (9) may alternatively be written as

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \sqrt{-g}j^\mu \,, \tag{15}$$

with $g = \det\,(g_{\mu\nu})$. Hint: the metric connection is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\} \,. \tag{16}$$

Furthermore, we have the identity

$$\partial_\mu g = g g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \,. \tag{17}$$

(2.4) Show that equation (9) implies that

$$\nabla_\mu j^\mu = 0 \,. \tag{18}$$

Question 3

Two observers, A and B, find themselves in free fall in orbits of constant r in the Schwarzschild metric (we take $c = 1$)

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{19}$$

The orbits are in the plane $\theta = \pi/2$ and have a radius $r_A = 4m, r_B = 4^{4/3}m$. At the time $t = 0$ the observers A and B both go through the point with $\phi = 0$.

For circular, timelike geodesics with radius r we have

$$(\dot{t})^2 = \frac{r}{r-3m} = (\dot{\phi})^2 r^3/m\,, \tag{20}$$

where the dot indicates differentiation with respect to the proper time.

(3.1) Calculate the coordinate time Δt_A that observer A needs for one revolution.

(3.2) How much time $\Delta\tau_A$ does observer A need for one revolution according to his own watch?

(3.3) The clock of observer B is lighted and can be read from a distance by observer A. What time difference $(\Delta\tau_B)'$ does A see at the watch of B between two successive passages of B through the point with $\phi = 0$? How much time $(\Delta\tau_A)'$ has evolved in the meantime according to his own watch?

(3.4) Observer A now uses his rocket motors to stop at the point with coordinates $r = 4m, \theta = \pi/2, \phi = 0$. He repeats the observations of question (2.3). What are now the results of these observations, i.e. what are now the values of $(\Delta\tau_B)'$ and $(\Delta\tau_A)'$?